Physics 618 2020

Classify U(1) Central extensions of Abelian Groups, Pontryagin Duality, Stone-von Neumann Theorem

April 21, 2020

Last time: We discussed a bit about the relation of a Lie group G to its Lie algebra $\mathfrak{g} = T_1 G$ but I neglected to state one of the main theorems in the subject:

$$\begin{tikzcd} G \arrow[r] & \text{Tangent space @ } 1 \\ & T_1 G \\ & \rotatebox{90}{$\in$} \\ \exp \xi & \xi \arrow[l] \end{tikzcd}$$

<u>Thm</u>

(a) Every finite dim'l Lie algebra $\mathfrak{g}$ over $k = \mathbb{R}$ is the Lie algebra of a <u>unique</u> (up to isom.) connected and <u>simply connected</u> Lie group.

(b.) $$\underset{\text{Lie group homom.}}{[f: G_1 \to G_2]} \Longleftrightarrow \underset{\text{Lie algebra homom.}}{[df : T_1 G_1 \to T_1 G_2]}$$

"Simple" Lie algebras : Nonabelian & no nontrivial ideals

$J \subset \mathfrak{g}$ sub Lie algebra

but $\forall\, X \in \mathfrak{g},\ Y \in J,\ [X, Y] \in J$

Example of nonsimple Lie algebra

Poincaré algebra = Lie algebra of Poincaré Lie group

or

Affine Euclidean

$$1 \to \mathbb{R}^d \to Euc(d) \to O(d) \to 1$$

Lie algebra Level : P_μ translations in μ^{th} direction

$M_{\mu\nu}$ rotations in $\mu\nu$ plane

$\downarrow$

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\lambda\rho}] = \delta_{\mu\lambda} P_\rho - \delta_{\mu\rho} P_\lambda$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \text{Sum of 4 } M\text{'s}$$

Simple Lie Group

(a.) It's Lie algebra is simple

$\Updownarrow$

(b.) G is connected, nonabelian and does not have nontrivial **Connected** normal subgroups.

N.B. SU(n) is a simple Lie group and

$$Z(SU(n)) = \left\{ \omega^j \mathbb{1}_n \;\middle|\; \omega = \exp \frac{2\pi i}{n} \right\}$$

$$\cong \mathbb{Z}_n$$

is a normal subgroup. So a "Simple Lie group" is not the same thing as a Lie group which is also a simple group.

Why you need the "connected and simply connected" criterion:

$$\underset{\text{Lie}}{\underset{\uparrow}{G}} \times \underset{\text{finite}}{\underset{\uparrow}{\Gamma}} \quad \text{same Lie algebra as } \underset{\text{Lie}}{\underset{\uparrow}{G}}$$

Similarly if $\Gamma \subset Z(G)$

$\rightarrow$ G/Γ is a Lie group with the same Lie algebra as G,

e.g.

$$\begin{tikzcd} SU(2)/\mathbb{Z}_2 \arrow[r, phantom, "\cong"] \arrow[d] & SO(3) \arrow[d] \\ su(2) \arrow[r, phantom, "\cong"] & so(3) \end{tikzcd}$$

su(2): 2×2 antihermitian traceless matrices

so(3): 3×3 real antisymmetric matrices

$$[J^i, J^j] = \epsilon^{ijk} J^k$$

Every compact Lie group

$$1 \longrightarrow \underbrace{G_0}_{\text{cpt of identity}} \longrightarrow G \longrightarrow \underbrace{\pi_0(G)}_{\text{group of components.}} \longrightarrow 1$$

$$G_0 = \frac{G_{ss} \times T^d}{\text{finite}}$$

$$G_{ss} = \prod_i G_i \leftarrow \text{compact simple.}$$

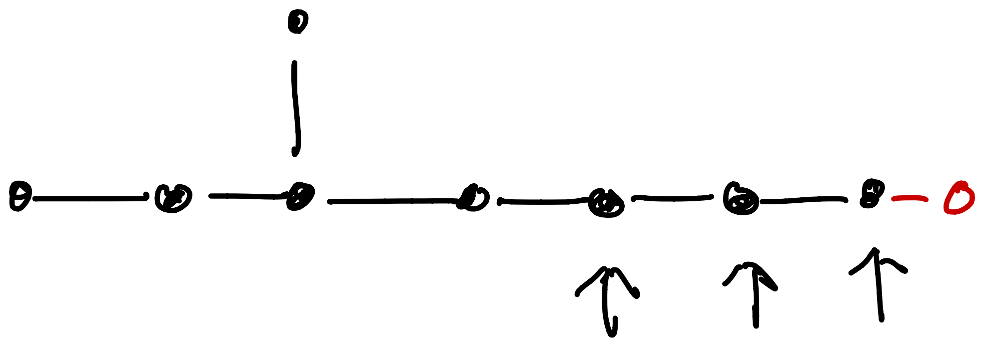
Compact simple, simply connected, connected have been classified by Cartan:

- $SU(n)$
- $SO(2n+1)$
- $USp(2n)$
- $SO(2n)$ (simply connected cover is $Spin(2n)$)

G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

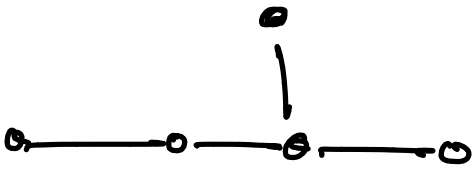
Dynkin diagrams E_n series

E_9 infinitedim'l.



E_8

"E_5"



$= D_5$.

$$\begin{tikzcd} D_5 \\ Spin(10) \arrow[u] \arrow[d] \\ SO(10) \end{tikzcd}$$

Heisenberg Groups

Consider general central extensions

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] & \widetilde{G} \arrow[r] & G \arrow[r] & 1 \\ & \text{Abelian} \arrow[u] & & \text{Abelian} \arrow[u] & \end{tikzcd}$$

$$\widetilde{G} = \{ (a,g) \mid a \in A, g \in G \}$$

$$(a_1, g_1) \cdot (a_2, g_2) = (a_1 a_2 \underset{\substack{\uparrow \\ \text{Cocycle.}}}{f}(g_1, g_2), g_1 g_2)$$

group commutator

$$\left[\left(\overset{\overset{Z(\widetilde{G})}{\downarrow}}{a_1}, g_1\right),(a_2,g_2)\right] = \left(\frac{f(g_1,g_2)}{f(g_2,g_1)}, \underset{\underset{G_{ab.}}{\uparrow}}{1}\right)$$

$k : G \times G \longrightarrow A$

"commutator function"

$$k = f(g_1,g_2)\, f(g_2,g_1)^{-1}$$

① invt under $f \sim f'$ by coboundary

② $\widetilde{G}$ Abelian iff $k = 1$

③ k skew, alternating, bimultiplicative

$$k(g_1 g_2, g_3) = k(g_1,g_3)\, k(g_2,g_3)$$

Def: General Heisenberg group is $\widehat{G}$ fits in such an extension with k nondegenerate i.e.

$g \neq 1$ then $\exists\, g'$ so that

$$k(g,g') \neq 1.$$

If k is nondegenerate then
$Z(\tilde{G}) \cong A$ i.e. $\tilde{G}$ is
a maximally non Abelian extension
of G by A.

$A = U(1)$ or a fin. gen.
subgroup of $U(1)$

G product of

- * Vector spaces
- * tori $\cong \mathbb{R}^d / \mathbb{Z}^d$
- * fin. generated Abelian groups

then we have a nice theorem:

Thm: Isom. classes of c.e. of such G by $U(1)$ are in 1-1 correspondence with continous bimultiplicative alternating functions $k : G \times G \to U(1)$

"alternating" : $k(g,g) = 1$

Much easier than solving cocycle relation and computing the coho. Class.

Remarks:

(1.) "Finitely Generated Abelian Groups"

$$A \approx \langle \underbrace{g_1, \cdots, g_n} \mid [g_i, g_j] = 1 \rangle$$

$$1 \to \underbrace{\mathrm{Tors}(A)}_{\text{finite Abelian}} \longrightarrow A \longrightarrow \mathbb{Z}^r \longrightarrow 1$$

Sequence will split, but not canonically.

def: For any Abelian group A there is a canonical finite subgroup

$$\mathrm{Tors}(A) := \left\{ a \in A \mid a \text{ has finite order} \right\}.$$

Important that A is Abelian.

$$A/\mathrm{Tors}(A) \cong \mathbb{Z}^r \quad \text{some } r = \text{"rank"}$$

So, what are the finite Abelian groups?

Kronecker Structure Theorem:

Every fin. Ab. gp. is isom. to a product of cyclic groups.

Noncanonically

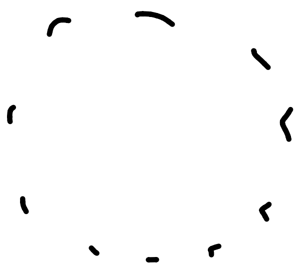
$$G \approx \left(\prod \mathbb{Z}/n_i\mathbb{Z}\right) \times \mathbb{Z}^d \times \left(\frac{\mathbb{R}^n}{\mathbb{Z}^n}\right) \times \bar{V}$$

Important that k only characterizes $\tilde{G}$ up to isomorphism. To construct an actual group one must find a cocycle f s.t. $k(g_1, g_2) = \frac{f(g_1, g_2)}{f(g_2, g_1)}$

Theorem guarantees that if k is bimult. and alternating, then f exists.

Pontryagin Duality

Recall discussion of clock & shift ops from Q.M. on



Def: S - Abelian group

Pontryagin dual group : $\hat{S} :=$ group of continuous homomorphisms

$$\chi : S \longrightarrow U(1)$$

"continuous" : The theory is richest for topological Abelian groups, in fact "locally compact topological Abelian groups"

X is loc. cpt. if $\forall\ x \in X$,

$\exists\ K \subset X$, Open $\mathcal{U}$ $x \in \mathcal{U} \subset K$.

↳ cpt.

finite groups } (discrete topology)
lattices }

tori, finite dimensional v.s. ↓ all loc. cpt. Abelian groups

∞-dim'l Hilbert space is a topological Abelian group, but is not loc. cpt.

S – loc. cpt. Abelian group

$$\hat{S} := \left\{ \begin{array}{l} \chi : G \to U(1) \text{ homom.} \\ \chi \text{ continuous} \end{array} \right\}$$

$\hat{S}$ is a locally compact Abelian group.

$$(\chi_1 \chi_2)(s) := \chi_1(s)\chi_2(s)$$

defines the group product.

Given $s \in S$ we have a map $\hat{S} \to U(1)$ be evaluation.

$$ev_s : \begin{array}{c}\chi\\ \rotatebox{90}{$\in$}\\ \hat{S}\end{array} \longrightarrow \chi(s) \in U(1)$$

[topologize $\hat{S}$ weakest top. so that ev_s is continuous]

$\hat{S}$ = Group of unitary 1-diml rep's of S'

S' Abllian of the above type all <u>irreps</u> are 1-diml.

$ev_s = \hat{s}$ is a character on the loc. cpt. Abelian group $\hat{S}$

$$\begin{array}{ccl} S & \longrightarrow & \hat{\hat{S}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s & \longrightarrow & \hat{s} : \chi \longrightarrow \chi(s). \end{array}$$

<u>Thm</u>: $$S \cong \hat{\hat{S}}$$

Examples 1: $S = \mathbb{Z}/n\mathbb{Z}$

$$\chi \in \mathrm{Hom}(S, U(1))$$

$$\chi(\bar{1}) \in U(1)$$

$$\chi(\bar{l}) = (\chi(\bar{1}))^{l} \qquad l = \bar{l} \text{ mod } n.$$

$$\chi(0) = \chi(\bar{n}) = (\chi(\bar{1}))^{n} = 1$$

$\therefore$ $\chi(\bar{1})$ is an n^{th} root of 1

$$\overbrace{\mathbb{Z}/n\mathbb{Z}} \cong \mathrm{Res}(n) \cong \mathbb{Z}/n\mathbb{Z}$$

Now by Kron. structure theorem we get P.D. for all fin. Abelian groups.

Example 2: $G = \mathbb{R}$ $\chi \in \widehat{\mathbb{R}}$

$$\chi(x+y) = \chi(x)\chi(y)$$

So $\chi(x) = \exp(ax)$ $a \in \mathbb{C}$

$\in U(1)$ $\underset{\overset{\shortparallel}{ik}}{a} \in i\mathbb{R}$. $k \in \mathbb{R}$

$$\chi_k(x) = e^{ikx} \qquad k \in \mathbb{R}$$

$$\widehat{\mathbb{R}} = \mathbb{R}.$$

Example 3: $S = \mathbb{Z} = \langle 1 \rangle$

$\xi = \chi(1) \in U(1)$

$$\chi_\xi(n) = \xi^n \qquad \chi_{\xi_1}\chi_{\xi_2} = \chi_{\xi_1\xi_2}$$

$$\widehat{\mathbb{Z}} \cong U(1)$$

Example 4 $S = U(1)$

$$\chi \in \mathrm{Hom}\Big(\underset{\underset{\mathbb{R}/\mathbb{Z}}{\parallel}}{U(1)}, U(1)\Big)$$

$$\mathbb{R} \longrightarrow U(1)$$

$x \in \mathbb{R}$, $\chi(x) = e^{ikx}$

$$= e^{ik(x+\mathbb{Z})} \qquad k \in \mathbb{Z}$$

$$\boxed{\chi_n(\xi) = \xi^n \qquad n \in \mathbb{Z}}$$

$\widehat{U(1)}$ 1-diml irreps of $U(1)$ are the integers.

$$\chi_{n_1}\chi_{n_2} = \chi_{n_1+n_2}.$$

$$\widehat{U(1)} \cong \mathbb{Z}$$

$$\widehat{\mathbb{Z}} \cong U(1).$$

Example 5

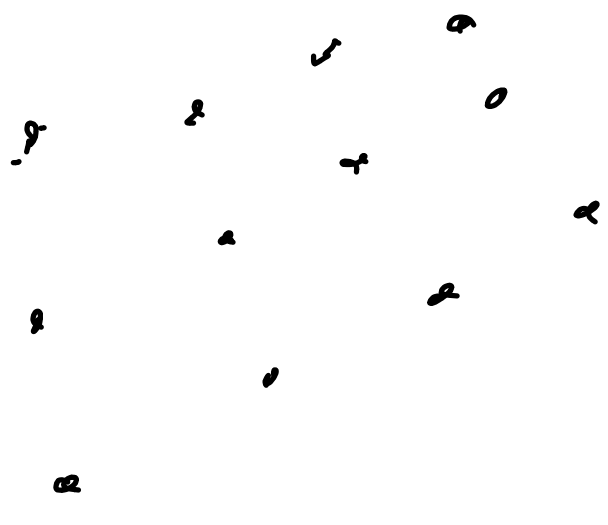
Generalizes

$$\widehat{\mathbb{Z}^d} = U(1)^d$$

$$\widehat{U(1)^d} = \mathbb{Z}^d$$

$G = \mathbb{Z}^d$ but embedded in Euclidean space $\mathbb{R}^d$

Lattice in $\mathbb{R}^d$ $\Gamma \subset \mathbb{R}^d$



$\in \mathbb{R}^2$

$$\mathbb{R}^d/\Gamma \cong \text{torus} \cong U(1)^d$$

$\Gamma \subset \mathbb{R}^d$ using Eucl. product can define dual lattice

$$\Gamma^\vee = \{ k \in \mathbb{R}^d \,/\, \underline{k \cdot \gamma} \in \mathbb{Z} \}$$

$\Gamma^\vee$ will also be a d-diml lattice.

What are the unitary irreps of Γ?

$$\chi_k(\gamma) = \exp(2\pi i\, k\cdot\gamma)$$

for any $\underline{k \in \mathbb{R}^d}$ we get a unitary irrep of Γ:

$$\chi_k(\gamma_1+\gamma_2) = \chi_k(\gamma_1)\chi_k(\gamma_2)$$

But this is a redundant parametrization.

$$k \qquad k+g \qquad g \in \Gamma^\vee$$

then $\chi_k = \chi_{k+g}$

$$\chi_{k+g}(\gamma) = \exp[2\pi i(k\cdot\gamma + \overset{\mathbb{Z}}{\overset{\downarrow}{g\cdot\gamma}})]$$

$$= \chi_k(\gamma)$$

So

$$P.D.(\Gamma) \cong \overset{\bar{k}}{\overset{\cap}{\mathbb{R}^d}} \Big/ \underset{\equiv}{\Gamma^\vee} \cong U(1)^d.$$

$$\chi_{\bar{k}}(\gamma) = \exp(2\pi i\, k\cdot\gamma)$$

where k is any lift of $\bar{k}$ to $\mathbb{R}^d$.

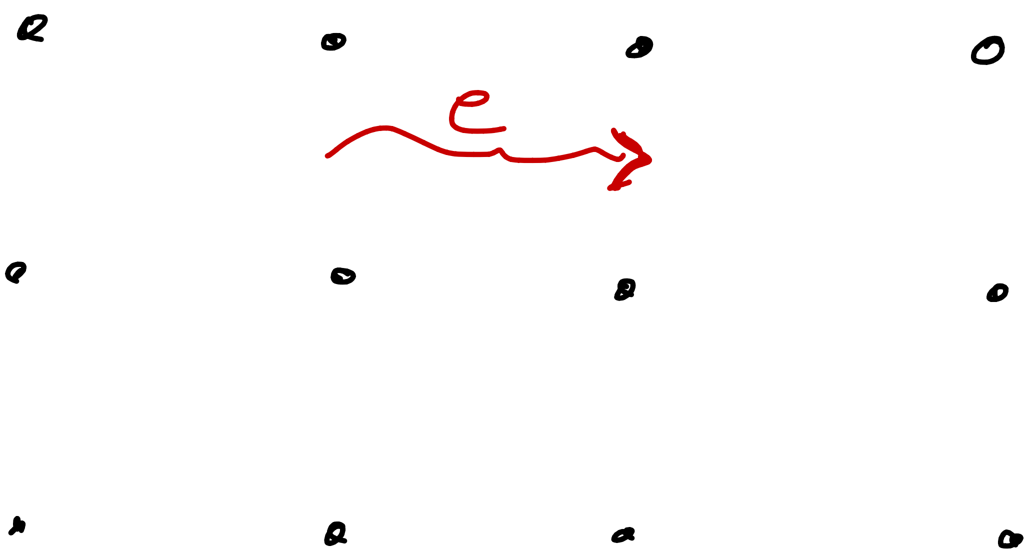
$$P.D.\left(\mathbb{R}^d/\Gamma^\vee\right) \cong \Gamma$$

$$\chi_\gamma(\bar{k}) = \exp\left(2\pi i\, k.\gamma\right)$$

* * *

Bloch's theorem in cond. matter physics.

Electrons moving in a crystal —
Regular array of charged atoms
$C \subset \mathbb{E}^d$ invariant under
translation by a lattice $\Gamma \subset \mathbb{R}^d$.



Reduce the Schrödinger problem of electrons in the crystal to that of a single electron moving through array of charged atoms.

$$\left(-\frac{\hbar^2}{2m}\nabla_x^2 + U(x)\right) = H$$

e.g. $$U(x) = \sum_{i\in C} \frac{-Z_i e}{|x - x_i|}$$

$$U(x+\gamma) = U(x) \quad \gamma \in \Gamma$$

So Γ commutes with H.

In this case can construct the translation operators $\boxed{T(\gamma)}$ act on $\mathcal{H}$.

So

$$1 \to \underline{\underline{\underline{U(1)}}} \longrightarrow \underbrace{U(1)\times\Gamma}_{\text{acts on Hilbert}} \longrightarrow \Gamma \to 0$$

Abelian group $\cong \Gamma$ commuting with H.

So: Eigenspaces of H decompose into irred. reps of Γ; those are 1 dim'l.

Labeled by $PD(\Gamma) = \underbrace{\mathbb{R}^d/\Gamma^\vee}$

"Brillouin torus."

$$[T(\gamma), H] = 0.$$

$$H\psi = E\psi$$

then $\psi(x) = e^{ik\cdot x} \underbrace{u_k(x)}_{\text{periodic under } \Gamma}$

$\frac{k}{2\pi} \in \Gamma^\vee$ = dual lattice

Solid state physics $2\pi\Gamma^\vee$ = "reciprocal lattice"

$u_k(x)$ periodic $x \longrightarrow x+\gamma$

descend to functions on

$\mathbb{R}^d/\Gamma$ compact torus.

Spectrum for a fixed k

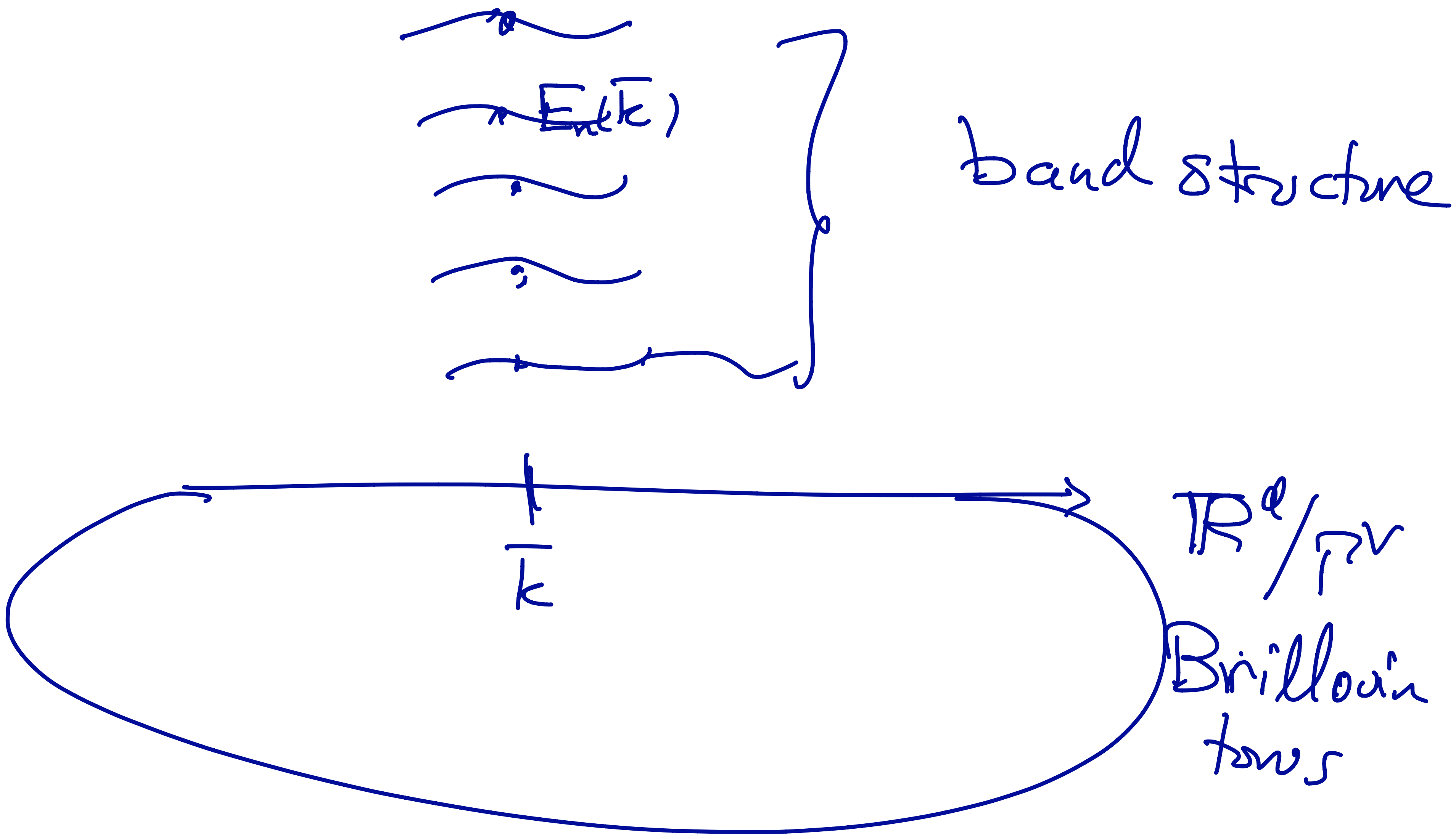
$e^{-ik\cdot x} H e^{ik\cdot x}$ = diff'l op. acting on functions on $T^d = \mathbb{R}^d/\Gamma$, but it depends on k

$$\overset{\shortparallel}{\left(-\frac{\hbar^2}{2m}\nabla_x^2 + \underset{\text{periodic}}{U(x)}\right)}$$

$$\longrightarrow H_k$$

H_k is an operator on functions on T^d/Γ = compact space

∴ has a discrete eigenspectrum.



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